Stratification of Tensor Triangular Categories

Applications to Motivic Categories

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Outline of Presentation

The Mathematical Landscape is Large

Categories $\mathcal{C}$ are pervasive in all fields of mathematics

- Algebra
  - $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$;
  - $\mathcal{C} = \mathrm{rep}(G)$;
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- Geometry
  - $\mathcal{C} = SmMan$;
  - $\mathcal{C} = Schemes$;
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Classification Goals: A mentsh trakht un Got lakht

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- Can't classify all finite dim representations of group G in positive characteristic case;
- Can't classify finite CW complexes up to homotopy equivalence;
- No more hope for classifying all complexes of sheaves on an algebraic variety V;
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- Technically after a classification the thick tensor ideals

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Let R be a commutative ring. There is a bijection

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$$\mathcal{C} \to \bigcup_{x\in\mathcal{C}} \mathrm{supph}(x)$$

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  - The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is the universal space with well-behaved notion of support

Classification Theorem and Examples

For mild assumptions on $\mathcal{K}$ there is a bijection

$$\{\text{Thick} \otimes\text{-ideals of } \mathcal{K}\} \xrightarrow{\sim} \{\text{Thomason Subsets of } Spc(\mathcal{K})\}$$
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- The large, non-compact objects, however, are very interesting!
  - Objects representing cohomology theories are not compact in SH
  - There are major open questions about the structure of the larger objects (e.g. The telescope conjecture)

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Balmer-Favi Support and Stratification

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- In recent work Barthel, Heard and Sanders (BHS, 2021) developed a support theory for noetherian large tt-categories $\mathcal{T}$
  - There is no "$\mathrm{Spc}(\mathcal{T})$" but can consider $\mathrm{Spc}(\mathcal{T}^c)$
  - The support for arbitrary objects will be a subset of $\mathrm{Spc}(\mathcal{T}^c)$

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When such a category admits this bijection, they say the category is ***stratified***.

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- D(X) for X noetherian scheme
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- Prove some longstanding fundamental conjectures in algebraic geometry (e.g. The Milnor conjecture and the Bloch-Kato conjecture).

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- There are invertible objects $R(n) \in DM(\mathbb{F}, R)$ for each $n \in \mathbb{Z}$, such that $R(n) \otimes R(m) = R(n+m)$ called the Tate Twists.

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The category $DM(\mathbb{F}, R)$ is extremely complex

- Idea: study first a "piece" of the category; the Tate motives
- The localizing subcategory generated by the Tate twists is the (large) category of Tate motives, denoted by $DTM(\mathbb{F}, R)$.

Étale Motives

There is a similar story to tell with so called étale motives.

- There is a (large) derived category of étale motives, denoted $\mathrm{DM}^{\text{ét}}(\mathbb{F}, R)$, again with an étale motive functor:

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These two constructions are very similar: there is an "étale sheafification" functor

$$\alpha_{ét} : DM(\mathbb{F}, R) \to DM^{ét}(\mathbb{F}, R)$$

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Moreover,

- Whenever $\mathbb{Q} \subset R$, $\alpha_{ét}$ is an equivalence of categories.

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

The Balmer spectrum of $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$

Theorem (Gallauer 2019)

(1) *The Balmer spectrum of* $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$ *is the following picture:*

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In this picture the specialization relations are pointing upwards.

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(2) $\mathrm{Spc}(\mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z})^c) \cong \mathrm{Spec}(\mathbb{Z})$

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(3) *The étale sheafification map induces a map* $\mathrm{Spec}(\mathbb{Z}) \xrightarrow{\mathrm{Spc}(\alpha_{\text{ét}})} \mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ *which is a homeomorphism onto the subspace* $\{\mathfrak{m}_o, e_p\}$

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

Remarks on These Computations

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}},\mathbb{Z})^c) = \begin{tikzcd}[row sep=small] m_2 \arrow[d, dash] & m_3 \arrow[d, dash] & & \cdots & m_p \arrow[d, dash] & \cdots \\ e_2 \arrow[drr, dash] & e_3 \arrow[dr, dash] & & \cdots & e_p \arrow[dll, dash] & \cdots \\ & & m_0 & & & \end{tikzcd}$$

Let us explain what these m_p, e_p, m_0 are:

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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Let us explain what these $\mathfrak{m}_p, e_p, \mathfrak{m}_0$ are:

$(\mathfrak{m}_p)$: kernal of map $\gamma_p^* : \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \to \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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(e_p): kernals of the composite

$$\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \xrightarrow{\gamma^*} \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c \xrightarrow{\alpha_{\text{ét}}} \mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$$

Again, this kernal coincides with those motives whose mod-p étale cohomology vanishes.

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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Let us explain what these $\mathfrak{m}_p, e_p, \mathfrak{m}_0$ are:

$(\mathfrak{m}_p)$: kernal of map $\gamma_p^* : \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \to \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$. A result of Gallauer is that this kernal coincides with those motives whose mod-p motivic cohomology vanishes.

(e_p): kernals of the composite

$$\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \xrightarrow{\gamma^*} \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c \xrightarrow{\alpha_{\acute{e}t}} \mathrm{DTM}^{\acute{e}t}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$$

Again, this kernal coincides with those motives whose mod-p étale cohomology vanishes.

$(\mathfrak{m}_0)$: kernal of the rationalization map $\gamma^* : \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \to \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Q})^c$, which coincides with those motives whose rational motivic cohomology vanishes.

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Remarks on These Computations

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{tikzcd}[row sep=small] \mathfrak{m}_2 \arrow[d, dash] & \mathfrak{m}_3 \arrow[d, dash] & \cdots & \mathfrak{m}_p \arrow[d, dash] & \cdots \\ \mathfrak{e}_2 \arrow[drr, dash] & \mathfrak{e}_3 \arrow[dr, dash] & \cdots & \mathfrak{e}_p \arrow[dl, dash] & \cdots \\ & & \mathfrak{m}_0 & & \end{tikzcd}$$

- Morally there are only 3 flavors of primes in $\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

Remarks on These Computations

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{tikzcd}[column sep=small, row sep=small]
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- Morally there are only 3 flavors of primes in $\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$
- Reduce the problem to each "vertical slice" in the spectrum and just consider the 3 primes in each slice

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

How To Establish Minimality at the Primes

$$
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\end{tikzcd}
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($\mathfrak{m}_p$): Suffices to pass to the "residue field" $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})$.

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- Gallauer proves that $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c \cong \mathrm{D}^b_{fil}(\mathbb{Z}/p\mathbb{Z})$ is equivalent to the filtered bounded derived category of $\mathbb{Z}/p\mathbb{Z}$

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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- Does this lift to the large categories?

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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- Gallauer showed the local categories at the primes $\mathfrak{e}_p$ are just $\mathrm{DTM}^{\acute{e}t}(\overline{\mathbb{Q}}, \mathbb{Z}_p)$

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}},\mathbb{Z})$

How To Establish Minimality at the Primes

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- Again pass to the residue field $\mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}},\mathbb{Z}/p\mathbb{Z})$. The Rigidity Theorem establishes stratification for this category.

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

How To Establish Minimality at the Primes

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(e_p): Have successfully shown minimality for these

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- Again pass to the residue field $\mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})$. The Rigidity Theorem establishes stratification for this category.

$(\mathfrak{m}_0)$: Reduces to showing minimality in $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Q})$.

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

Final Comments and Summery

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{tikzcd}[column sep=small] m_2 \arrow[d, dash] & m_3 \arrow[d, dash] & \cdots & m_p \arrow[d, dash] & \cdots \\ e_2 \arrow[drr, dash] & e_3 \arrow[dr, dash] & \cdots & e_p \arrow[dl, dash] & \cdots \\ & & m_0 & & \end{tikzcd}$$

- In summary, we want to get a classification for the localizing tensor ideals for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$.

My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

Final Comments and Summery

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- In summary, we want to get a classification for the localizing tensor ideals for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$.
- Using the results of Sanders, et al we are tasked with checking a certain minimality condition at every prime.

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My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

Final Comments and Summery

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- In summary, we want to get a classification for the localizing tensor ideals for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$.
- Using the results of Sanders, et al we are tasked with checking a certain minimality condition at every prime.
- In this case, we can first take vertical slices of the spectrum, and then check minimality at local categories for mod p and rational coefficients