

# Stratification of Tensor Triangular Categories

## Applications to Motivic Categories

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- 1 **Balmer Support and the Classification Theorem**

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- 2 **Stratification**

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- 3 **Derived Categories of Motives**

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- 4 **My Research Problem**

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## The Mathematical Landscape is Large

Categories  $\mathcal{C}$  are pervasive in all fields of mathematics

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- Geometry
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Unfortunately, we are confronted with "wild classification problems"

- Can't classify all finite dim representations of group  $G$  in positive characteristic case;
- Can't classify finite CW complexes up to homotopy equivalence;
- No more hope for classifying all complexes of sheaves on an algebraic variety  $V$ ;
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- Technically after a classification the thick tensor ideals

## Some Historical Examples

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$$\{\text{Thick } \otimes\text{-ideals of } \text{stab}(kG)\} \xrightarrow{\sim} \{\text{Specialization Closed subsets of } \text{Proj}(H^\bullet(G, k))\}$$
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- The pair  $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$  is the universal space with well-behaved notion of support

## Classification Theorem and Examples

For mild assumptions on  $\mathcal{K}$  there is a bijection

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  - There are major open questions about the structure of the larger objects (e.g. The telescope conjecture)

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- In recent work Barthel, Heard and Sanders (BHS, 2021) developed a support theory for noetherian large tt-categories  $\mathcal{T}$ 
  - There is no " $\text{Spc}(\mathcal{T})$ " but can consider  $\text{Spc}(\mathcal{T}^c)$
  - The support for arbitrary objects will be a subset of  $\text{Spc}(\mathcal{T}^c)$

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- This theory greatly increased our understanding of motivic cohomology.
- Prove some longstanding fundamental conjectures in algebraic geometry (e.g. The Milnor conjecture and the Bloch-Kato conjecture).

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- There are invertible objects  $R(n) \in DM(\mathbb{F}, \mathbb{R})$  for each  $n \in \mathbb{Z}$ , such that  $R(n) \otimes R(m) = R(n + m)$  called the Tate Twists.

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- The motivic cohomology groups for a variety  $X$  are then defined as

$$H^{m,n}(X) = \text{hom}_{DM(\mathbb{F}, \mathbb{R})}(R(X), R(n)[m])$$

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- Idea: study first a "piece" of the category; the Tate motives
- The localizing subcategory generated by the Tate twists is the (large) category of Tate motives, denoted by  $DTM(\mathbb{F}, \mathbb{R})$ .

## Étale Motives

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- Whenever  $\mathbb{Q} \subset \mathbb{R}$ ,  $\alpha_{\text{ét}}$  is an equivalence of categories.

# My Problem: Stratification for $\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

The Balmer spectrum of  $\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$

Theorem (Gallauer 2019)

(1) *The Balmer spectrum of  $\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$  is the following picture:*

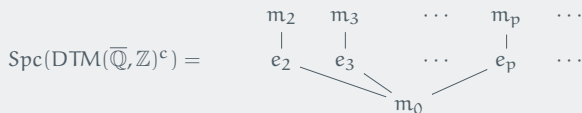


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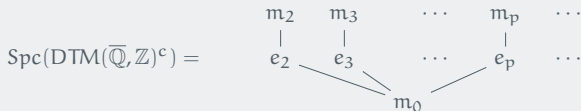
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(3) *The étale sheafification map induces a map  $\text{Spec}(\mathbb{Z}) \xrightarrow{\text{Spc}(\alpha_{\text{ét}})} \text{Spc}(\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$  which is a homeomorphism onto the subspace  $\{m_0, e_p\}$*

# My Problem: Stratification for $\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## Remarks on These Computations

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$(e_p)$ : kernels of the composite

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- Morally there are only 3 flavors of primes in  $\text{Spc}(\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$
- Reduce the problem to each "vertical slice" in the spectrum and just consider the 3 primes in each slice

# My Problem: Stratification for $\text{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## How To Establish Minimality at the Primes

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## Final Comments and Summary

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- In this case, we can first take vertical slices of the spectrum, and then check minimality at local categories for mod  $p$  and rational coefficients